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Periodic solutions of a class of degenerate parabolic system with delays ☆

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ABSTRACT

This paper is concerned with a class of periodic degenerate parabolic system with time delays in a bounded domain under mixed boundary condition. Under locally Lipschitz condition on reaction functions, we apply Schauder fixed point theorem to obtain the existence of periodic solutions of the periodic problem. With quasi-monotonicity in addition, we also show that the periodic problem has a maximal and a minimal periodic solutions. Applications of the obtained results are also given to some nonlinear diffusion models arising from ecology.

1. Introduction

Periodic semi-linear reaction diffusion equations are of particular interests since they can take into account seasonal fluctuations occurring in the phenomena appearing in the models, and have been extensively studied by many researchers (of [1,2,4,7,12,17,24–26,28,29]). At the same time, much attentions have also been paid to periodic semi-linear parabolic equations with delays where time lag is taken into consideration in nonlinear reaction functions (of [11,21,27,31,33–35]). In order to model a more general way of spreading behavior in space, the use of degenerate parabolic operators has been proposed. For instance, replacing the usual $-\Delta u$ term by a degenerate elliptic operator as $-\Delta u^m$ is a way to model the diffusion of species that dislike crowding, see [15,25]. In this paper, we are concerned with the quasi-linear parabolic system with delays of degenerate type

$$\frac{\partial u_i}{\partial t} = \Delta\varphi_i(u_i) + f_i(x,t,\mathbf{u},\mathbf{u}_\tau), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$\mathcal{B}_i u = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

$$u_i(x,t) = u_i(x,t+\omega), \quad x \in \Omega,\ -\tau_i \leqslant t \leqslant 0,\ i = 1,\dots,p, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ the Laplacian with respect to the space variable $x = (x_1,\dots,x_N)$, and $\mathcal{B}_i$ $(i = 1,\dots,p)$ are given by

$$\mathcal{B}_i u_i = \frac{\partial u_i}{\partial \nu}, \quad x \in \Gamma_1,\ t > 0, \tag{1.4}$$

$$\mathcal{B}_i u_i = u_i, \quad x \in \Gamma_2,\ t > 0 \tag{1.5}$$

with $\frac{\partial}{\partial\nu}$ denoting the outward normal derivative on Γ_1, $\Gamma_1, \Gamma_2 \neq \emptyset$ and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, (1.2) is the mixed boundary condition which appears in a rather natural way in some ecological models (of [3] and references therein),

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$$\mathbf{u} \equiv \mathbf{u}(x,t) = \big(u_1(x,t), \dots, u_p(x,t)\big),$$
$$\mathbf{u}_\tau \equiv \mathbf{u}_\tau(x,t) = \left(\int_{t-\tau_1}^{t} J_1(x,t-s)u_1(x,s)\,ds, \dots, \int_{t-\tau_p}^{t} J_p(x,t-s)u_p(x,s)\,ds\right)$$

with the integral kernels J_i and delays $\tau_i > 0$. For each $i = 1, \dots, p$, $f_i(x,t,\cdot)$ is Hölder continuous and periodic in time variable t with a fixed period $\omega > 0$, $\varphi_i(u)$ will be precisely prescribed in following section and $\varphi_i(u) = u^m$ $(m > 1)$ is the special case. Such problems are of interest in reaction diffusion processes in biology and chemistry. For example, the case for single equations

$$\frac{\partial u}{\partial t} = \Delta u^m + u(x,t)\left(a(x) - b(x)u - \int_{t-\tau}^{t} J(x,t-s)u(x,s)\,ds\right), \tag{1.6}$$

was considered by Badii et al. (of [6]), which generalizes the classical Volterra's population equation in that it includes space dependence and nonlinear diffusion effects. To investigate the attractivity of ω-periodic solutions of (1.1)–(1.3), we also consider the system (1.1)(1.2) with initial value condition

$$u_i(x,t) = \eta_i(x,t), \quad x \in \Omega,\ -\tau_i \leqslant t \leqslant 0, \tag{1.7}$$

where $\eta_i(x,t)$ $(i = 1, \dots, p)$ are nonnegative functions.

Degenerate parabolic equations model nonlinear diffusion phenomena and have been the subject of extensive study, and most of the works are devoted to the existence, uniqueness, regularity properties of generalized solutions and some interesting properties such as the finite speed of propagation of perturbation, localization and blow up (of [3,8,9,20,30,34] and references therein). Recently periodic degenerate parabolic equations have also received considerable attention (of [5,18, 19,22,32,35]), in particular, the existence of periodic solutions of the Nicholson's blowflies model with delays was established in [35] by constructing some suitable Lyapunov functionals and applying Leray–Schauder fixed point theorem. However, the literature about periodic problems for degenerate parabolic systems is scarce, see recent papers [13,14] where Fragnelli et al. considered the existence of nontrivial periodic solutions of degenerate parabolic systems with delayed nonlocal terms and Dirichlet boundary conditions via the topological degree theory, and to our knowledge, it appears that little discussion is devoted to the coupled system of degenerate parabolic equations with delays under mixed boundary condition.

In this paper, we mainly investigate the existence of periodic solutions of problem (1.1)–(1.3) without any quasi-monotone condition on reaction functions. Quite different from the semi-linear delayed diffusion equations, Eq. (1.1) takes the nonlinear degenerate diffusion into account, which causes some difficulties in the proof of the existence and uniqueness of solutions of the related periodic problem. To overcome the encountered difficulties, we exploit some ideas in dealing with the initial boundary problem for degenerate parabolic equations.

The paper is organized as follows. In Section 2, we give the definitions of upper and lower periodic solutions of problem (1.1)–(1.3) and apply Schauder fixed point theorem to find the periodic solutions between appropriate upper and lower solutions for problem (1.1)–(1.3) without any quasi-monotone condition on reaction functions. In Section 3, for quasi-monotone nondecreasing reaction functions, we use the method of monotone iteration to show the existence of a maximal and a minimal periodic solutions of problem (1.1)–(1.3). The monotone iteration schemes involve only uncoupled equations and are potentially useful for the computation of numerical solutions. In final section, as the application of main results in previous section the sufficient conditions are given for the existence of nontrivial nonnegative periodic solutions of the delayed logistic equation with the nonlinear diffusion, and so called coexistence periodic solutions of the delayed degenerate parabolic system which models the competitor–competitor–mutualist system, respectively. In the case of linear diffusion, these models were studied by many researchers (of [10–12,15–17,27,28,37]).

2. Existence of periodic solutions

Let

$$D = \Omega \times (0,\infty), \qquad \Gamma = \partial\Omega \times (0,\infty), \tag{2.1}$$
$$Q_0^{(i)} = \overline{\Omega} \times [-\tau_i, 0], \qquad Q_0 = \prod_{i=1}^{p} Q_0^{(i)}, \tag{2.2}$$
$$Q^{(i)} = \overline{\Omega} \times [-\tau_i, \infty), \qquad Q = \prod_{i=1}^{p} Q^{(i)}. \tag{2.3}$$

Denoted by $C^\alpha(D^*)$ the set of functions in $C(D^*)$ that are α-Hölder continuous with exponent $\alpha \in (0,1)$, where D^* is any one of the above domains. The set of vector functions $\mathbf{u} \equiv (u_1, \dots, u_p)$ of the above space are denoted by $\mathcal{C}^\alpha(D^*)$. Similar notations are defined for other function spaces.

Throughout this paper we make the following assumptions:

(H1) $\varphi_i \in C^1([0,\infty)) \cap C^2((0,\infty))$, $\varphi_i(0) = \varphi_i'(0) = 0$ and $\varphi_i'(s) > 0$ for all $s \neq 0$;

(H2) $f_i(x,t,\cdot) \in C^\alpha(\overline{D})$, $f_i(\cdot,\mathbf{u},\mathbf{v})$ satisfies the local Lipschitz condition, i.e. there exists constant k_i such that for $(\mathbf{u},\mathbf{v}), (\mathbf{u}',\mathbf{v}') \in \wedge \times \wedge_\tau$ given below

$$\left|f_i(\cdot,\mathbf{u},\mathbf{v}) - f_i\left(\cdot,\mathbf{u}',\mathbf{v}'\right)\right| \leqslant k_i\left(\left|\mathbf{u}-\mathbf{u}'\right| + \left|\mathbf{v}-\mathbf{v}'\right|\right)$$

with $|\mathbf{u}| = \sum_{i=1}^p |u_i|$ for vector $\mathbf{u} = (u_1,\dots,u_p) \in \mathbb{R}^p$;

(H3) The integral kernel $J_i(x,t)$ is nonnegative, piece continuous in t, Hölder continuous in x uniformly in t and possesses the property

$$\int_0^{\tau_i} J_i(x,t)\,dt = 1, \quad x \in \Omega.$$

Since $\varphi_i'(0) = 0$, the equation in system (1.1) is degenerate parabolic and problem (1.1)(1.2)(1.7) usually admits solutions only in some generalized sense. Following e.g. [3,23], we introduce the following definitions.

Definition 2.1. A vector function $\mathbf{u}(x,t) = (u_1(x,t),\dots,u_p(x,t))$ is said to be the solution of problem (1.1)(1.2)(1.7) on $\Omega_T = \Omega \times (0,T)$, if

(i) $u_i \in C([0,T]; L^1(\Omega)) \cap L^\infty(\Omega_T)$, $\varphi_i(u_i) \in L^2((0,T); H^1(\Omega))$.
(ii) $u_i(x,t) = \eta_i(x,t)$ for $t \in [-\tau_i, 0]$.
(iii) For any $\psi_i \in C^{2,1}(\overline{\Omega}_T)$ $(i=1,\dots,p)$ which vanishes for $(x,t) \in \Gamma_1 \times (0,T)$ and every $t \in [0,T]$,

$$\int_\Omega u_i(x,t)\psi_i(x,t)\,dx + \int_0^t\int_{\Gamma_1} \frac{\partial\psi_i}{\partial\nu}\varphi_i(u_i)\,dx\,ds$$
$$= \int_\Omega \eta_i(x,0)\psi_i(x,0)\,dx + \int_0^t\int_\Omega \left(u_i\frac{\partial\psi_i}{\partial s} + f_i(x,s,\mathbf{u},\mathbf{u}_\tau)\psi_i + \varphi_i(u_i)\Delta\psi_i\right)dx\,ds.$$

A pair of coupled upper and lower solutions of problem (1.1)–(1.3) is defined as follows.

Definition 2.2. A pair of functions $\tilde{\mathbf{u}} = (\tilde{u}_1,\dots,\tilde{u}_p)$, $\hat{\mathbf{u}} = (\hat{u}_1,\dots,\hat{u}_p)$ in $\mathcal{C}(Q) \cap \mathcal{L}^2((0,\omega); H^1(\Omega))$ are said to be coupled upper and lower solutions of problem (1.1)–(1.3) if $\tilde{\mathbf{u}} \geqslant \hat{\mathbf{u}}$,

$$\tilde{u}_i(x,t+\omega) \leqslant \tilde{u}_i(x,t), \qquad \hat{u}_i(x,t+\omega) \geqslant \hat{u}_i(x,t), \quad (x,t) \in Q_0^{(i)}$$

and for every $(\mathbf{u},\mathbf{v})$ in $\wedge \times \wedge_\tau$,

$$\int_\Omega \tilde{u}_i(x,t)\psi_i(x,t)\,dx + \int_0^t\int_{\Gamma_1} \frac{\partial\psi_i}{\partial\nu}\varphi_i(\tilde{u}_i)\,dx\,ds$$
$$\geqslant \int_\Omega \tilde{u}_i(x,0)\psi_i(x,0)\,dx + \int_0^t\int_\Omega \left(\tilde{u}_i\frac{\partial\psi_i}{\partial s} + f_i(x,s,\mathbf{u},\mathbf{v})\psi_i + \varphi_i(\tilde{u}_i)\Delta\psi_i\right)dx\,ds$$

when $u_i = \tilde{u}_i$,

$$\int_\Omega \hat{u}_i(x,t)\psi_i(x,t)\,dx + \int_0^t\int_{\Gamma_1} \frac{\partial\psi_i}{\partial\nu}\varphi_i(\hat{u}_i)\,dx\,ds$$
$$\leqslant \int_\Omega \hat{u}_i(x,0)\psi_i(x,0)\,dx + \int_0^t\int_\Omega \left(\hat{u}_i\frac{\partial\psi_i}{\partial s} + f_i(x,s,\mathbf{u},\mathbf{v})\psi_i + \varphi_i(\hat{u}_i)\Delta\psi_i\right)dx\,ds$$

when $u_i = \hat{u}_i$, where nonnegative function $\psi_i \in C^{2,1}(\overline{D})$ $(i=1,\dots,p)$ and vanishes for $(x,t) \in \Gamma_1 \times (0,T)$, and

$$\wedge = \{\mathbf{u} \in \mathcal{C}(\overline{D});\ \hat{\mathbf{u}} \leqslant \mathbf{u} \leqslant \tilde{\mathbf{u}}\},$$
$$\wedge_\tau = \{\mathbf{v} \in \mathcal{C}(Q);\ \hat{\mathbf{u}}_\tau \leqslant \mathbf{v} \leqslant \tilde{\mathbf{u}}_\tau\}.$$

We will apply Schauder fixed point theorem to obtain the existence of periodic solutions of problem (1.1)–(1.3) when reaction functions do not possess any quasi-monotone property.

Theorem 2.1. *Let $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ be a pair of coupled upper and lower solutions of problem* (1.1)–(1.3). *Then problem* (1.1)–(1.3) *admits at least one ω-periodic solution $\mathbf{u} \in \wedge$.*

The following lemma plays a crucial role in the proof of the above theorem.

Lemma 2.1. *For given $\mathbf{w} = (w_1, \ldots, w_p) \in \wedge$ with $\mathbf{w}(x, t+\omega) = \mathbf{w}(x,t)$, the following periodic parabolic problem*

$$\frac{\partial u_i}{\partial t} - \Delta\varphi_i(u_i) + k_i u_i = k_i w_i + f_i(x, t, \mathbf{w}, \mathbf{w}_\tau), \quad x \in \Omega,\ t > 0, \tag{2.4}$$
$$\mathcal{B}_i u_i = 0, \quad x \in \partial\Omega,\ t > 0, \tag{2.5}$$
$$u_i(x,t) = u_i(x, t+\omega), \quad x \in \Omega,\ t \geqslant 0 \tag{2.6}$$

has a unique periodic solution $u_i \in \{w \in C(\overline{D});\ \hat{u}_i(x,t) \leqslant w(x,t) \leqslant \tilde{u}_i(x,t),\ w(x,t) = w(x, t+\omega)\}$ $(i = 1, \ldots, p)$.

Proof. We will use the Poincaré operator method to prove that problem (2.4)–(2.6) admits a periodic solution $u_i \in \wedge_{i\omega} = \{w \in C(\overline{D});\ \hat{u}_i(x,t) \leqslant w(x,t) \leqslant \tilde{u}_i(x,t),\ w(x,t) = w(x,t+\omega)\}$. To this end, we define a Poincaré map $T: C(\overline{\Omega}) \to C(\overline{\Omega})$ by $T(u_{i0}(x)) = u_i(x,\omega)$, where $u_i(x,t)$ is the solution of the initial boundary value problem (2.4)(2.5) with the initial condition

$$u_i(x,0) = u_{i0}(x), \quad x \in \Omega. \tag{2.7}$$

By the result of [3,23], the map T is well defined.

Let $u_i(x,t)$ be the solution of the initial boundary value problem (2.4)(2.5)(2.7) with $u_{i0}(x) = \tilde{u}_i(x,0)$.

From Definition 2.2 and the Lipschitz continuity of f_i, it follows that $k_i\tilde{u}_i + f_i(x,t,\mathbf{u},\mathbf{v}) \geqslant k_i w_i + f_i(x,t,\mathbf{w},\mathbf{w}_\tau)$ with $u_i = \tilde{u}_i$ and thus for any $t \in (0, \omega]$

$$\begin{aligned}&\int_\Omega \big(u_i(x,t) - \tilde{u}_i(x,t)\big)\psi_i(x,t)\,dx + \int_0^t \int_{\Gamma_1} \frac{\partial\psi_i}{\partial\nu}\big(\varphi_i(u_i) - \varphi_i(\tilde{u}_i)\big)\,dx\,ds \\ &\quad\leqslant \int_0^t \int_\Omega \left\{(u_i - \tilde{u}_i)\left(\frac{\partial\psi_i}{\partial s} + k_i\psi_i\right) + \big(\varphi_i(u_i) - \varphi_i(\tilde{u}_i)\big)\Delta\psi_i\right\}dx\,ds.\end{aligned} \tag{2.8}$$

Then we can get $u_i(x,\omega) \leqslant \tilde{u}_i(x,\omega)$ by the arguments in the proof of comparison results in [3,23]. Indeed, define

$$\eta_i(x,t) = \begin{cases} \frac{\varphi_i(u_i) - \varphi_i(\tilde{u}_i)}{u_i - \tilde{u}_i}, & u_i \neq \tilde{u}_i, \\ 0, & \text{otherwise}. \end{cases}$$

Let $\{\eta_{ik}\}_{k=1}^\infty$ be sufficiently smooth and uniformly bounded such that

$$\eta_{ik} \geqslant \frac{1}{k}, \qquad \lim_{k\to\infty} \big\|(\eta_{ik} - \eta_i)/\sqrt{\eta_{ik}}\big\|_{L^2(\Omega\times(0,t))} = 0.$$

Consider the following adjoint boundary problem for any $k \geqslant 1$ and $\lambda > 0$,

$$\frac{\partial\psi_i}{\partial s} + \eta_{ik}\Delta\psi_i = \lambda\psi_i, \quad x \in \Omega,\ s \in (0,t),$$
$$\mathcal{B}_i\psi_i = 0, \quad x \in \partial\Omega,\ s \in (0,t),$$
$$\psi_i(x,t) = \chi_i(x), \quad x \in \overline{\Omega}.$$

By Lemma 4.1 of [3], this problem has a unique solution ψ_{ik}, which is smooth in $\Omega \times (0,t)$ for any smooth $\chi_i \in (0,1)$, and

$$0 \leqslant \psi_{ik} \leqslant C, \qquad \iint_{\Omega\times(0,t)} \eta_{ik}(\Delta\psi_{ik})^2\,dx\,ds \leqslant C,$$

where C is the positive constant independent of k.

Taking test function $\psi_i = \psi_{ik}$ in (2.8), the solution of the above adjoint boundary problem with $\chi_i = \operatorname{sgn}(u_i(x,t) - \tilde{u}_i(x,t))^+$ and $\eta^+ = \max\{0, \eta\}$, we have

$$\int_\Omega \big(u_i(x,t) - \tilde{u}_i(x,t)\big)^+ \,dx \leqslant C\left[\int_0^t \int_\Omega (u_i - \tilde{u}_i)^+ \,dx\,ds + \left\| \frac{\eta_{ik} - \eta_i}{\sqrt{\eta_{ik}}} \right\|_{L^2(\Omega\times(0,t))} \int_0^t \int_\Omega \eta_{ik} (\Delta\psi_{ik})^2 \,dx\,ds\right].$$

Thence letting $k \to \infty$ and using Gronwall's inequality, we have $u_i(x,t) \leqslant \tilde{u}_i(x,t)$.

Therefore by Definition 2.2, we have $u_i(x,\omega) \leqslant \tilde{u}_i(x,\omega) \leqslant \tilde{u}_i(x,0)$, and then an induction argument leads to

$$\cdots \leqslant u_i(x, n\omega) \leqslant \cdots \leqslant u_i(x, 2\omega) \leqslant u_i(x,\omega) \leqslant \tilde{u}_i(x,\omega) \leqslant \tilde{u}_i(x,0) = u_{i0}(x).$$

By the definition of map T, it is observed that $T^n(u_{i0}) = T(T^{n-1}(u_{i0})) = u_i(x, n\omega)$, $n = 1, 2, \ldots,$ and $\{T^n(\tilde{u}_i(x,0))\}$ is the monotone nonincreasing sequence.

Similarly, by considering the initial boundary value problem (2.4)(2.5)(2.7) with $u_{i0}(x) = \hat{u}_i(x,0)$, we can also get the monotone nondecreasing sequence $\{T^n(\hat{u}_i(x,0))\}$. Furthermore, we have $\hat{u}_i \leqslant T^n(\hat{u}_i(x,0)) \leqslant T^n(\tilde{u}_i(x,0)) \leqslant \tilde{u}_i$, $n = 1, \ldots$ by an induction argument.

Therefore by the rather standard arguments in [36], we can get the maximal periodic solution and the minimal periodic solution of problem (2.4)–(2.6) in $\wedge_{i\omega}$.

In what follows, we show the uniqueness of periodic solutions of problem (2.4)–(2.6). Assume that $u^*(x,t)$ and $u_*(x,t)$ are the periodic solutions of problem (2.4)–(2.6).

Denoting $k_i w_i + f_i(x, t, \mathbf{w}, \mathbf{w}_\tau)$ by $h_i(x,t)$ and considering $u^*_\epsilon(x,t)$ and $u_{*\epsilon}(x,t)$ as the solutions of the initial boundary value problem

$$\begin{aligned}
&\frac{\partial u_i}{\partial t} - \Delta\big(\varphi_i(u_i) + \epsilon u_i\big) + k_i u_i = h_i(x,t), \quad x \in \Omega,\ t > 0,\\
&\mathcal{B}_i u_i = 0, \quad x \in \partial\Omega,\ t > 0
\end{aligned} \tag{2.9}$$

with the initial value $u^*(x,0)$ and $u_*(x,0)$ respectively, we have $\lim_{\epsilon\to 0} |u^*_\epsilon(x,t) - u^*(x,t)| = 0$, $\lim_{\epsilon\to 0} |u_{*\epsilon}(x,t) - u_*(x,t)| = 0$ for $(x,t) \in \overline{\Omega} \times [\omega, 2\omega]$ and $\varphi_i(u^*_\epsilon)$, $\varphi_i(u_{*\epsilon}) \in L^2((0,\omega); H^1(\Omega))$. Indeed, multiplying Eq. (2.9) by $\varphi_i(u^*_\epsilon)$ and integrating over $Q_\omega = \Omega \times (\omega, 2\omega)$, we obtain

$$\iint_{Q_\omega} \varphi_i\big(u^*_\epsilon\big) \frac{\partial u^*_\epsilon}{\partial t} \,dx\,dt + \iint_{Q_\omega} \big|\nabla\varphi_i\big(u^*_\epsilon\big)\big|^2 \,dx\,dt \leqslant \iint_{Q_\omega} \big\{-k_i u^*_\epsilon + h_i(x,t)\big\} \varphi_i\big(u^*_\epsilon\big) \,dx\,dt,$$

and thus

$$\iint_{Q_\omega} \big|\nabla\varphi_i\big(u^*_\epsilon\big)\big|^2 \,dx\,dt \leqslant C,$$

where C is the constant independent of ϵ.

Multiplying Eq. (2.9) by $\psi(x,t) = \operatorname{sgn}_\delta(\varphi_i(u^*_\epsilon) - \varphi_i(u_{*\epsilon}))$ where $\operatorname{sgn}_\delta(s)$ is the smooth approximation function of $\operatorname{sgn}(s)$ on $\mathbb{R}$ with $\operatorname{sgn}_\delta(s) = 1$ for $s \geqslant \delta$, $\operatorname{sgn}_\delta(s) = -1$ for $s \leqslant -\delta$ and $\operatorname{sgn}'_\delta(s) \geqslant 0$, and integrating over Q_ω, we can get

$$\begin{aligned}
&\iint_{Q_\omega} \left[\frac{\partial(u^*_\epsilon - u_{*\epsilon})}{\partial t} + k_i\big(u^*_\epsilon - u_{*\epsilon}\big)\right] \operatorname{sgn}_\delta\big(\varphi_i\big(u^*_\epsilon\big) - \varphi_i(u_{*\epsilon})\big) \,dx\,dt\\
&\qquad + \iint_{Q_\omega} \nabla\big(u^*_\epsilon - u_{*\epsilon}\big) \nabla \operatorname{sgn}_\delta\big(\varphi_i\big(u^*_\epsilon\big) - \varphi_i(u_{*\epsilon})\big) \,dx\,dt\\
&\quad = -\iint_{Q_\omega} \big|\nabla\varphi_i\big(u^*_\epsilon\big) - \nabla\varphi_i(u_{*\epsilon})\big|^2 \operatorname{sgn}'_\delta\big(\varphi_i\big(u^*_\epsilon\big) - \varphi_i(u_{*\epsilon})\big) \,dx\,dt\\
&\quad \leqslant 0.
\end{aligned} \tag{2.10}$$

Thence letting $\delta \to 0$ in (2.10), we have

$$\int_\Omega \big|u^*_\epsilon(x, 2\omega) - u_{*\epsilon}(x, 2\omega)\big| \,dx + k_i \iint_{Q_\omega} \big|u^*_\epsilon(x,t) - u_{*\epsilon}(x,t)\big| \,dx\,dt \leqslant \int_\Omega \big|u^*_\epsilon(x,\omega) - u_{*\epsilon}(x,\omega)\big| \,dx. \tag{2.11}$$

Furthermore we can arrive at

$$\iint\limits_{Q_\omega} \left|u^*(x,t) - u_*(x,t)\right| dx\,dt \leqslant 0$$

by letting $\epsilon \to 0$ in (2.11) and the periodicity of u^* and u_*, which implies that $u^*(x,t) = u_*(x,t)$. □

Proof of Theorem 2.1. The existence of periodic solutions of problem (1.1)–(1.3) will be proven by means of the Schauder fixed point theorem, for which we consider the following periodic problem

$$\begin{aligned}
&\frac{\partial u_i}{\partial t} - \Delta\varphi_i(u_i) + k_i u_i = q_i(x,t), \quad x \in \Omega,\ t > 0,\\
&\mathcal{B}_i(u_i) = 0, \quad x \in \partial\Omega,\ t > 0,\\
&u_i(x,t) = u_i(x, t+\omega), \quad x \in \Omega,\ t \geqslant 0,
\end{aligned} \tag{2.12}$$

where $q_i \in C_\omega(\overline{D}) = \{w \in C(\overline{D});\ \ w(x,t) = w(x,t+\omega)\}$. From the proof of Lemma 2.1, it follows that problem (2.12) has the unique solution $u_i \in C_\omega(\overline{D})$, which is denoted by $S_i q_i$. It can be inferred that the operator $S_i : C_\omega(\overline{D}) \to C_\omega(\overline{D})$ is compact and continuous by the similar discussion in [19,32]. Now if $F_i(\mathbf{u})(x,t) = k_i u_i(x,t) + f_i(x,t,\mathbf{u}(x,t),\mathbf{u}_\tau(x,t))$, $\mathcal{F}(\mathbf{u}) = (F_1(\mathbf{u}),\dots,F_p(\mathbf{u}))$, and $\mathcal{S}\circ\mathcal{F} = (S_1F_1,\dots,S_pF_p)$, then the fixed point of operator $\mathcal{S}\circ\mathcal{F}$ is the periodic solution of problem (1.1)–(1.3).

Let $\mathcal{X}$ be the closed bounded convex subset given by

$$\mathcal{X} = \left\{\mathbf{u} \in \mathcal{C}(\overline{D});\ \hat{\mathbf{u}} \leqslant \mathbf{u} \leqslant \tilde{\mathbf{u}} \text{ and } \mathbf{u}(x,t) = \mathbf{u}(x,t+\omega) \text{ in } D\right\},$$

then it follows that $\mathcal{S}\circ\mathcal{F}$ map $\mathcal{X}$ into itself from Lemma 2.1. Therefore by the Schauder fixed point theorem, $\mathcal{S}\circ\mathcal{F}$ has at least one fixed point $\mathbf{u} \in \mathcal{X}$, which is the solution of problem (1.1)–(1.3). □

3. Mixed quasi-monotone functions

In this section we use the method of monotone iteration to show the existence of periodic solutions of problem (1.1)–(1.3) under some quasi-monotone conditions on reaction function

$$\mathbf{f}(x,t,\mathbf{u},\mathbf{u}_\tau) \equiv \left(f_1(x,t,\mathbf{u},\mathbf{u}_\tau),\dots,f_p(x,t,\mathbf{u},\mathbf{u}_\tau)\right).$$

By writing vectors $\mathbf{u},\mathbf{u}_\tau \in \mathbb{R}^p$ in the split forms

$$\mathbf{u} \equiv \left(u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}\right), \qquad \mathbf{u}_\tau \equiv \left([\mathbf{u}_\tau]_{c_i}, [\mathbf{u}_\tau]_{d_i}\right),$$

where a_i, b_i, c_i, d_i are some nonnegative integers and $[\mathbf{u}]_\sigma$ denotes a vector with σ components of $[\mathbf{u}]$, we have the following definitions.

Definition 3.1. (See [27].) A vector function $\mathbf{f}(\cdot,\mathbf{u},\mathbf{v})$ is said to be mixed quasi-monotone in $\wedge\times\wedge_\tau$ if for each $i = 1,\dots,p$, there exist nonnegative integers a_i, b_i, c_i, d_i with $a_i + b_i = p-1$, $c_i + d_i = p$, such that $f_i(\cdot,\mathbf{u},\mathbf{v})$ is monotone nondecreasing in $[\mathbf{u}]_{a_i}$, $[\mathbf{v}]_{c_i}$ and is monotone nonincreasing in $[\mathbf{u}]_{b_i}$, $[\mathbf{v}]_{d_i}$ for all $(\mathbf{u},\mathbf{v}) \in \wedge\times\wedge_\tau$. The function $\mathbf{f}(\cdot,\mathbf{u},\mathbf{v})$ is said to be quasi-monotone nondecreasing in $\wedge\times\wedge_\tau$ if $b_i = d_i = 0$.

Definition 3.2. For mixed quasi-monotone function $\mathbf{f}(\cdot,\mathbf{u},\mathbf{v})$ in $\wedge\times\wedge_\tau$, ω-periodic functions $\bar{\mathbf{u}}$, $\underline{\mathbf{u}}$ are said to be a pair of periodic quasi-solutions of problem (1.1)–(1.3) if the following equalities are satisfied in weak sense

$$\begin{aligned}
&\frac{\partial \bar{u}_i}{\partial t} = \Delta\varphi_i(\bar{u}_i) + f_i\left(x,t,\bar{u}_i,[\bar{\mathbf{u}}]_{a_i},[\underline{\mathbf{u}}]_{b_i},[\bar{\mathbf{u}}_\tau]_{c_i},[\underline{\mathbf{u}}_\tau]_{d_i}\right), \quad x \in \Omega,\ t > 0,\\
&\frac{\partial \underline{u}_i}{\partial t} = \Delta\varphi_i(\underline{u}_i) + f_i\left(x,t,\underline{u}_i,[\underline{\mathbf{u}}]_{a_i},[\bar{\mathbf{u}}]_{b_i},[\underline{\mathbf{u}}_\tau]_{c_i},[\bar{\mathbf{u}}_\tau]_{d_i}\right),\\
&\mathcal{B}_i\bar{u}_i = 0, \quad x \in \partial\Omega,\ t > 0,\\
&\mathcal{B}_i\underline{u}_i = 0, \quad i = 1,\dots,p.
\end{aligned}$$

From Definition 2.2 and Definition 3.1, it follows that upper and lower solutions are generally coupled. However, for quasi-monotone nondecreasing vector function $\mathbf{f}(\cdot,\mathbf{u},\mathbf{v})$, upper and lower solutions are not coupled, and we first consider problem (1.1)–(1.3) in this situation.

Theorem 3.1. *Let $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ be a pair of the upper and lower solutions of problem* (1.1)–(1.3) *and assume that $\mathbf{f}(\cdot,\mathbf{u},\mathbf{v})$ is quasi-monotone nondecreasing in $\wedge\times\wedge_\tau$, then there exist a maximal periodic solution $\bar{\mathbf{u}}(x,t)$ and a minimal periodic solution $\underline{\mathbf{u}}(x,t)$ of problem* (1.1)–(1.3) *in $\wedge$ in the sense that if $\mathbf{u}(x,t)$ is any other ω-periodic solution of problem* (1.1)–(1.3) *in $\wedge$ then $\underline{\mathbf{u}}(x,t) \leqslant \mathbf{u}(x,t) \leqslant \bar{\mathbf{u}}(x,t)$.*

Moreover, if $\mathbf{u}(x,t;\eta)$ *is the solution of the initial boundary problem* (1.1), (1.2), (1.7) *with* $\eta(x,t) = (\eta_1(x,t), \dots, \eta_p(x,t)) \in \mathcal{S}_0 = \{\eta(x,t);\ \hat{u}_i(x,t) \leqslant \eta_i(x,t) \leqslant \tilde{u}_i(x,t) \text{ for } (x,t) \in Q_0^{(i)}\}$*, then for any* $\varepsilon > 0$*, there exists* $t^* > 0$ *such that*

$$\underline{u}_i(x,t) - \varepsilon \leqslant u_i(x,t;\eta) \leqslant \bar{u}_i(x,t) + \varepsilon, \quad i = 1, \dots, p \tag{3.1}$$

for $x \in \overline{\Omega}$ *and* $t \geqslant t^*$.

Proof. Similar to the proofs of Theorem A and Theorem B in Ref. [27], the maximal periodic solution $\bar{\mathbf{u}}(x,t)$ and minimal periodic solution $\underline{\mathbf{u}}(x,t)$ of (1.1)–(1.3) are the limit functions of so called maximal sequence $\{\bar{\mathbf{u}}^k\}$ and minimal sequence $\{\underline{\mathbf{u}}^k\}$ respectively, which are constructed by the iteration scheme

$$\begin{aligned}
&\frac{\partial \bar{u}_i^{k+1}}{\partial t} - \triangle\varphi_i\big(\bar{u}_i^{k+1}\big) + k_i\bar{u}_i^{k+1} = k_i\bar{u}_i^k + f_i\big(x,t,\bar{\mathbf{u}}^k,\bar{\mathbf{u}}_\tau^k\big), \quad x \in \Omega,\ t > 0,\\
&\mathcal{B}_i\bar{u}_i^{k+1} = 0, \quad x \in \partial\Omega,\ t > 0,\\
&\bar{u}_i^{k+1}(x,t) = \bar{u}_i^k(x,t+\omega), \quad x \in \Omega,\ -\tau_i \leqslant t \leqslant 0,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\frac{\partial \underline{u}_i^{k+1}}{\partial t} - \triangle\varphi_i\big(\underline{u}_i^{k+1}\big) + k_i\underline{u}_i^{k+1} = k_i\underline{u}_i^k + f_i\big(x,t,\underline{\mathbf{u}}^k,\underline{\mathbf{u}}_\tau^k\big), \quad x \in \Omega,\ t > 0,\\
&\mathcal{B}_i\underline{u}_i^{k+1} = 0, \quad x \in \partial\Omega,\ t > 0,\\
&\underline{u}_i^{k+1}(x,t) = \underline{u}_i^k(x,t+\omega), \quad x \in \Omega,\ -\tau_i \leqslant t \leqslant 0,
\end{aligned} \tag{3.3}$$

where $\bar{\mathbf{u}}^0(x,t) = \tilde{\mathbf{u}}(x,t)$, $\underline{\mathbf{u}}^0(x,t) = \hat{\mathbf{u}}(x,t)$.

In fact, by Definition 2.2, Definition 3.1 and (3.2) with $k = 0$, we have

$$\begin{aligned}
&\frac{\partial \tilde{u}_i}{\partial t} - \triangle\varphi_i(\tilde{u}_i) + k_i\tilde{u}_i \geqslant \frac{\partial \bar{u}_i^1}{\partial t} - \triangle\varphi_i\big(\bar{u}_i^1\big) + k_i\bar{u}_i^1, \quad x \in \Omega,\ t > 0,\\
&\mathcal{B}_i\tilde{u}_i \geqslant \mathcal{B}_i\bar{u}_i^1, \quad x \in \partial\Omega,\ t > 0,\\
&\tilde{u}_i(x,0) \geqslant \bar{u}_i^1(x,0), \quad x \in \Omega,
\end{aligned}$$

and thus $\bar{u}_i^1 \leqslant \tilde{u}_i$ by comparison theorem in [3]. Similarity, we have $\hat{u}_i \leqslant \underline{u}_i^1$, $i = 1, \dots, p$.

On the other hand, by (3.2), (3.3) with $k = 0$, the Lipschitz continuity and monotone property of f_i, we have

$$\begin{aligned}
&\frac{\partial \bar{u}_i^1}{\partial t} - \triangle\varphi_i\big(\bar{u}_i^1\big) + k_i\bar{u}_i^1 \geqslant \frac{\partial \underline{u}_i^1}{\partial t} - \triangle\varphi_i\big(\underline{u}_i^1\big) + k_i\underline{u}_i^1, \quad x \in \Omega,\ t > 0,\\
&\mathcal{B}_i\bar{u}_i^1 \geqslant \mathcal{B}_i\underline{u}_i^1, \quad x \in \partial\Omega,\ t > 0,\\
&\bar{u}_i^1(x,0) \geqslant \underline{u}_i^1(x,0), \quad x \in \Omega,
\end{aligned}$$

and thus $\underline{u}_i^1 \leqslant \bar{u}_i^1$ by comparison theorem in [3] again.

Thence we get

$$\hat{\mathbf{u}} \leqslant \underline{\mathbf{u}}^1 \leqslant \bar{\mathbf{u}}^1 \leqslant \tilde{\mathbf{u}}.$$

An induction argument, using the monotone property of f_i, shows that

$$\hat{\mathbf{u}} \leqslant \underline{\mathbf{u}}^k \leqslant \underline{\mathbf{u}}^{k+1} \leqslant \bar{\mathbf{u}}^{k+1} \leqslant \bar{\mathbf{u}}^k \leqslant \tilde{\mathbf{u}}.$$

Therefore by the regularity arguments of degenerate parabolic equations as that in [19], we can conclude that there exist functions $\bar{\mathbf{u}}(x,t) = (\bar{u}_1(x,t), \dots, \bar{u}_p(x,t))$ and $\underline{\mathbf{u}}(x,t) = (\underline{u}_1(x,t), \dots, \underline{u}_p(x,t))$ such that as $k \to \infty$,

$$\nabla\varphi_i\big(\bar{u}_i^k\big) \rightharpoonup \nabla\varphi_i(\bar{u}_i), \qquad \nabla\varphi_i\big(\underline{u}_i^k\big) \rightharpoonup \nabla\varphi_i(\underline{u}_i) \quad \text{in } L^2\big(\Omega \times (0,2\omega)\big) \tag{3.4}$$

and

$$\bar{u}_i^k \to \bar{u}_i, \qquad \underline{u}_i^k \to \underline{u}_i \quad \text{in } C\big(\overline{\Omega} \times [0,2\omega]\big), \tag{3.5}$$

which are the maximal periodic solution and the minimal periodic solution of problem (1.1)–(1.3), respectively.

In fact, we first get the periodicity of $\bar{u}_i(x,t)$ by (3.5) and $\bar{u}_i^{k+1}(x,0) = \bar{u}_i^k(x,\omega)$. Next it follows from (3.2) that for any $\psi \in C^{2,1}(\overline{\Omega}_T)$ which vanishes for $(x,t) \in \Gamma_1 \times (0,T)$ and every $t \in [0,T]$,

$$\int_\Omega \bar{u}_i^{k+1}(x,t)\psi(x,t)\,dx + \int_0^t\int_{\Gamma_1} \frac{\partial\psi}{\partial\nu}\varphi_i\big(\bar{u}_i^{k+1}\big)\,dx\,ds + k_i\int_0^t\int_\Omega \bar{u}_i^{k+1}\psi\,dx\,ds$$
$$= \int_\Omega \bar{u}_i^k(x,\omega)\psi(x,0)\,dx + k_i\int_0^t\int_\Omega \bar{u}_i^k\psi\,dx\,ds + \int_0^t\int_\Omega\left(\bar{u}_i^{k+1}\frac{\partial\psi}{\partial s} + f_i\big(x,s,\bar{\mathbf{u}}^k,\bar{\mathbf{u}}_\tau^k\big)\psi + \varphi_i\big(\bar{u}_i^{k+1}\big)\Delta\psi\right)dx\,ds.$$

So by (3.5), the periodicity of $\bar{u}_i(x,t)$ and taking $k\to\infty$ in the above equality, we can see that $\bar{\mathbf{u}}(x,t)$ is the periodic solution of problem (1.1)–(1.3). Similarly $\underline{\mathbf{u}}(x,t)$ is proved to be the periodic solution of problem (1.1)–(1.3). Finally let $\mathbf{u}^*(x,t)=(u_1^*(x,t),\ldots,u_p^*(x,t))$ be the ω-periodic solution of problem (1.1)–(1.3) in $\wedge$. Then since $\mathbf{f}(\cdot,\mathbf{u},\mathbf{v})$ is quasi-monotone nondecreasing in $\wedge\times\wedge_\tau$, by Definition 2.2, the consideration of $(\mathbf{u}^*,\tilde{\mathbf{u}})$ and $(\hat{\mathbf{u}},\mathbf{u}^*)$ as the pair of upper and lower solutions in the above argument leads to $\underline{\mathbf{u}}\leqslant\mathbf{u}^*\leqslant\bar{\mathbf{u}}$.

Now we turn to prove (3.1). Denote $u_i\wedge\tilde{u}_i=\min(u_i,\tilde{u}_i)$, $u_i\vee\tilde{u}_i=\max(u_i,\tilde{u}_i)$, $\mathbf{u}\wedge\tilde{\mathbf{u}}=(u_1\wedge\tilde{u}_1,\ldots,u_p\wedge\tilde{u}_p)$, $\mathbf{u}\vee\tilde{\mathbf{u}}=(u_1\vee\tilde{u}_1,\ldots,u_p\vee\tilde{u}_p)$ and let $u_i^*(x,t;\eta)$ $(i=1,\ldots,p)$ be the solution of the initial value problem

$$\begin{aligned}
&\frac{\partial u_i^*}{\partial t}-\triangle\varphi_i\big(u_i^*\big)=f_i\big(x,t,\big(\mathbf{u}^*\wedge\tilde{\mathbf{u}}\big)\vee\hat{\mathbf{u}},\big(\mathbf{u}_\tau^*\wedge\tilde{\mathbf{u}}_\tau\big)\vee\hat{\mathbf{u}}_\tau\big),\quad x\in\Omega,\ t>0,\\
&\mathcal{B}_iu_i^*=0,\quad x\in\partial\Omega,\ t>0,\\
&u_i^*(x,t)=\eta_i(x,t),\quad x\in\Omega,\ -\tau_i\leqslant t\leqslant 0.
\end{aligned}\tag{3.6}$$

Then since $\eta(x,t)=(\eta_1(x,t),\ldots,\eta_p(x,t))\in\mathcal{S}_0$ and the monotone nondecreasing property of $f(\cdot,\mathbf{u},\mathbf{v})$, the comparison theorem of degenerate parabolic equations [3] leads to $\hat{u}_i(x,t)\leqslant u_i^*(x,t)\leqslant\tilde{u}_i(x,t)$, and thus $\mathbf{u}(x,t;\eta)=\mathbf{u}^*(x,t;\eta)$ by the uniqueness of the solution of (1.2)(1.3)(1.7).

Combining (3.6) with (3.2), we can get that $u_i(x,t+\omega;\eta)\leqslant\bar{u}_i^1(x,t)$ for $x\in\overline{\Omega}$, $t>0$ by the comparison theorem. An induction argument shows that $u_i(x,t+k\omega;\eta)\leqslant\bar{u}_i^k(x,t)$. Similarly we can get $\underline{u}_i^k(x,t)\leqslant u_i(x,t+k\omega;\eta)$. Therefore noting that (3.5) and letting $k\to\infty$ in $\underline{u}_i^k(x,t)\leqslant u_i(x,t+k\omega;\eta)\leqslant\bar{u}_i^k(x,t)$, (3.1) holds. □

For problem (1.1)–(1.3) with the mixed quasi-monotone reaction functions, we can get the following result.

Theorem 3.2. *Let $\tilde{\mathbf{u}},\hat{\mathbf{u}}$ be a pair of upper and lower solutions of* (1.1)–(1.3) *and assume that $\mathbf{f}(\cdot,\mathbf{u},\mathbf{v})$ is mixed quasi-monotone in $\wedge\times\wedge_\tau$, then there exists a pair of periodic quasi-solutions $\bar{\mathbf{u}}(x,t)$, $\underline{\mathbf{u}}(x,t)$ of* (1.1)–(1.3) *in $\wedge$. Moreover, if $\mathbf{u}(x,t;\eta)$ is the solution of the initial boundary problem* (1.1)(1.2)(1.7) *with $\eta(x,t)=(\eta_1(x,t),\ldots,\eta_p(x,t))\in\mathcal{S}_0=\{\eta(x,t);\ \hat{u}_i(x,t)\leqslant\eta_i(x,t)\leqslant\tilde{u}_i(x,t)$ for $(x,t)\in Q_0^{(i)}\}$, then for any $\varepsilon>0$, there exists $t^*>0$ such that*

$$\underline{u}_i(x,t)-\varepsilon\leqslant u_i(x,t;\eta)\leqslant\bar{u}_i(x,t)+\varepsilon,\quad i=1,\ldots,p\tag{3.7}$$

for $x\in\overline{\Omega}$ and $t\geqslant t^*$.

If $\bar{\mathbf{u}}(x,t)=\underline{\mathbf{u}}(x,t)$, then there exists unique periodic solution $\mathbf{u}^*(x,t)(=\underline{\mathbf{u}}(x,t))$ in $\wedge$ satisfying

$$\lim_{t\to\infty}\max_{x\in\Omega}\big|\mathbf{u}(x,t;\eta)-\mathbf{u}^*(x,t)\big|=0.\tag{3.8}$$

Proof. By letting $v_i=M-u_i$, $\mathbf{v}=\mathbf{M}-\mathbf{u}$ with constant vector $\mathbf{M}=(M_1,\ldots,M_p)$ satisfying $\mathbf{M}\geqslant\tilde{\mathbf{u}}$, we imbed system (1.1), (1.2) into an extended $2p$ system

$$\frac{\partial u_i}{\partial t}=\Delta\varphi_i(u_i)+F_i(x,t,\mathbf{u},\mathbf{v},\mathbf{u}_\tau,\mathbf{v}_\tau),\quad x\in\Omega,\ t>0,\tag{3.9}$$
$$\frac{\partial v_i}{\partial t}=\sum_{j=1}^N\frac{\partial}{\partial x_j}\left(\varphi_i'(M_i-v_i)\frac{\partial v_i}{\partial x_j}\right)+G_i(x,t,\mathbf{u},\mathbf{v},\mathbf{u}_\tau,\mathbf{v}_\tau),\tag{3.10}$$
$$\frac{\partial u_i}{\partial\nu}=0,\qquad\frac{\partial v_i}{\partial\nu}=0,\quad x\in\Gamma_1,\ t>0,\tag{3.11}$$
$$u_i=0,\qquad v_i=M_i,\quad x\in\Gamma_2,\ t>0,\ i=1,\ldots,p,\tag{3.12}$$

where

$$\begin{aligned}
&F_i(x,t,\mathbf{u},\mathbf{v},\mathbf{u}_\tau,\mathbf{v}_\tau)=f_i\big(x,t,u_i,[\mathbf{u}]_{a_i},[\mathbf{M}-\mathbf{v}]_{b_i},[\mathbf{u}_\tau]_{c_i},[\mathbf{M}-\mathbf{v}_\tau]_{d_i}\big),\\
&G_i(x,t,\mathbf{u},\mathbf{v},\mathbf{u}_\tau,\mathbf{v}_\tau)=-f_i\big(x,t,M_i-v_i,[\mathbf{M}-\mathbf{v}]_{a_i},[\mathbf{u}]_{b_i},[\mathbf{M}-\mathbf{v}_\tau]_{c_i},[\mathbf{u}_\tau]_{d_i}\big).
\end{aligned}$$

If $\mathbf{w}=(\mathbf{u},\mathbf{v})$, $\mathbf{w}_\tau=(\mathbf{u}_\tau,\mathbf{v}_\tau)$, it is easy to see that

$$\big(F_1(\cdot,\mathbf{w},\mathbf{w}_\tau),\dots,F_p(\cdot,\mathbf{w},\mathbf{w}_\tau),G_1(\cdot,\mathbf{w},\mathbf{w}_\tau),\dots,G_p(\cdot,\mathbf{w},\mathbf{w}_\tau)\big)$$

is quasi-monotone nondecreasing in $\wedge^* \times \wedge_\tau^*$, where $\wedge^* = \{\mathbf{w} \in \mathcal{C}(\bar D) \times \mathcal{C}(\bar D);\ (\hat{\mathbf{u}}, \mathbf{M} - \tilde{\mathbf{u}}) \leqslant \mathbf{w} \leqslant (\tilde{\mathbf{u}}, \mathbf{M} - \hat{\mathbf{u}})\}$ and $\wedge_\tau^* = \{\mathbf{z} \in \mathcal{C}(\bar Q) \times \mathcal{C}(\bar Q);\ (\hat{\mathbf{u}}_\tau, \mathbf{M} - \tilde{\mathbf{u}}_\tau) \leqslant \mathbf{z}_\tau \leqslant (\tilde{\mathbf{u}}_\tau, \mathbf{M} - \hat{\mathbf{u}}_\tau)\}$. Therefore by using Theorem 3.1, we can proceed as in the proof of Theorem 4.2 in [25] to prove the theorem. Indeed, let $\tilde{\mathbf{w}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$, $\hat{\mathbf{w}} = (\hat{\mathbf{u}}, \hat{\mathbf{v}})$ with $\tilde{\mathbf{v}} = \mathbf{M} - \hat{\mathbf{u}}$, $\hat{\mathbf{v}} = \mathbf{M} - \tilde{\mathbf{u}}$. Then by Definition 2.2 and the quasi-monotone nondecreasing property of $(F_1(\cdot,\mathbf{w},\mathbf{w}_\tau),\dots,F_p(\cdot,\mathbf{w},\mathbf{w}_\tau),G_1(\cdot,\mathbf{w},\mathbf{w}_\tau),\dots,G_p(\cdot,\mathbf{w},\mathbf{w}_\tau))$ in $\wedge^* \times \wedge_\tau^*$, the pair $\tilde{\mathbf{w}} = (\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$, $\hat{\mathbf{w}} = (\hat{\mathbf{u}}, \hat{\mathbf{v}})$ are order upper and lower solutions of (3.9)–(3.12). Similarly to the proof of Theorem 3.1, we can conclude that there exist the maximal periodic solution $\bar{\mathbf{w}} = (\bar{\mathbf{u}}, \mathbf{M} - \underline{\mathbf{u}})$ and minimal periodic solution $\underline{\mathbf{w}} = (\underline{\mathbf{u}}, \mathbf{M} - \bar{\mathbf{u}})$ of (3.9)–(3.12). By Definition 3.2, $(\bar{\mathbf{u}}, \underline{\mathbf{u}})$ is a pair of periodic quasi-solutions of problem (1.1)–(1.3).

Since the relation $\eta_i^* = M_i - \eta_i$ implies $M_i - \tilde u_i \leqslant \eta_i^* \leqslant M_i - \hat u_i$ for any $\eta \in \mathcal{S}_0$, the initial function (η, η^*) satisfies that $\hat{\mathbf{w}} \leqslant (\eta, \eta^*) \leqslant \tilde{\mathbf{w}}$, thence (3.7), (3.8) can be proved by the argument as that for (3.1). □

4. Applications

In this section, as the application of the existence results in the previous section we consider some periodic degenerate parabolic equations with delays, which includes a delayed logistic equation and a delayed competitor–competitor–mutualist system.

4.1. A delayed logistic equation with nonlinear diffusion

Consider the logistic diffusion equation with delay

$$\frac{\partial u}{\partial t} - \Delta u^m = u\big(a(x,t) - bu - cu_\tau\big), \quad x \in \Omega,\ t > 0, \tag{4.1}$$

under the boundary condition

$$\mathcal{B}u = 0, \quad x \in \partial\Omega,\ t > 0, \tag{4.2}$$

where $u_\tau(x,t) = u(x, t-\tau)$ with $\tau > 0$, $m > 1$, b, c are positive constants, function $a(x,t)$ is ω-periodic in time t and changes sign in Ω for any time t. The equation of this type models the evolution of the biological species living in the periodic environment, and the nonlinear diffusion term Δu^m ($m > 1$) was proposed in [15] in order to represent the tendency of the species to avoid crowding.

In the case of $m = 1$, Eq. (4.1) reduces to be semi-linear parabolic equation and has been investigated by many authors (of [11,27,37]). In this situation, the sufficient conditions to ensure the existence of nontrivial periodic solutions are usually given in terms of the periodic parabolic principal eigenvalue. If $c = 0$, Eq. (4.1) is the porous medium equation without delay, and the existence of nontrivial periodic solutions of the periodic problem was shown in [19].

For ω-periodic continuous function $f(x,t)$, we denote $f_M = \max_{\Omega \times [0,\omega]} f(x,t)$.

Theorem 4.1. *Let $b > c$ and assume that*

$$\Omega_+ \equiv \left\{ x \in \Omega;\ \frac{1}{\omega} \int_0^\omega a(x,t)\,dt > \frac{c}{b} a_M \right\} \neq \emptyset,$$

then there exists nontrivial nonnegative periodic solution $u^*(x,t)$ of (4.1), (4.2)*.*

Proof. From $\Omega_+ \neq \emptyset$ and the continuity of $a(x,t)$, it follows that there exist a ball $B_\rho(x_0) \subset \Omega_+$ and $\delta > 0$ such that

$$\frac{1}{\omega} \int_0^\omega a(x,t)\,dt \geqslant \frac{c}{b} a_M + \delta, \quad \text{for } x \in \overline{B_\rho(x_0)}.$$

Let ψ be eigenfunction corresponding first eigenvalue μ_1 of the eigenvalue problem

$$-\Delta\psi = \mu\psi, \quad x \in B_\rho(x_0),$$
$$\psi = 0, \quad x \in \partial B_\rho(x_0),$$

with $\|\psi\|_{L^\infty} = 1$.

Set $\underline{a}(t) = \min_{B_\rho(x_0)}\{a(x,t)\} - \frac{c}{b} a_M$, $\alpha(t) = \underline{a}(t) - \frac{1}{\omega}\int_0^\omega \underline{a}(t)\,dt$, and define $\tilde u(x,t) = \frac{a_M}{b}$,

$$\hat u(x,t) = \begin{cases} \varepsilon \exp\left(\int_0^t \alpha(s)\,ds\right) \psi^{\frac{1}{m}}(x), & (x,t) \in B_\rho(x_0) \times \mathbb{R}^+, \\ 0, & (x,t) \in (\Omega \setminus B_\rho(x_0)) \times \mathbb{R}^+. \end{cases}$$

Noting $\frac{\partial\psi}{\partial\nu} \leqslant 0$ for $x \in \partial B_\rho(x_0)$, it can be verified that $\tilde{u}(x,t)$ and $\hat{u}(x,t)$ are the ordered periodic upper and lower solutions of problem (4.1), (4.2), provided that

$$\varepsilon < \min\left\{\left(\frac{\delta}{2\mu_1}\right)^{\frac{1}{m-1}}, \frac{\delta}{2b}, \frac{a_M}{b}\right\} \min_{t\in[0,\omega]} \exp\left(\int_t^0 \alpha(s)\,ds\right).$$

Therefore by Theorem 2.1, problem (4.1), (4.2) admits at least one solution $u^*(x,t)$ satisfying $\hat{u}(x,t) \leqslant u^*(x,t) \leqslant \tilde{u}(x,t)$. □

4.2. *A competitor–competitor–mutualist system with delay*

Consider a periodic degenerate parabolic system with delay which involves interactions among a competitor, a competitor and a competitor–mutualist. The system of equations is given by

$$\frac{\partial u_1}{\partial t} - \Delta u_1^{m_1} = u_1\left(\alpha_1(x,t) - \frac{u_1}{a_1(x,t)} - \frac{a_2(x,t)u_{2\tau}}{1+a_3(x,t)u_{3\tau}}\right), \quad x \in \Omega,\ t > 0, \tag{4.3}$$

$$\frac{\partial u_2}{\partial t} - \Delta u_2^{m_2} = u_2\left(\alpha_2(x,t) - b_1(x,t)u_{1\tau} - \frac{u_2}{b_2(x,t)}\right), \tag{4.4}$$

$$\frac{\partial u_3}{\partial t} - \Delta u_3^{m_3} = u_3\left(\alpha_3(x,t) - \frac{u_3}{c_0(x,t)+c_1(x,t)u_{1\tau}}\right), \tag{4.5}$$

with the boundary condition

$$\mathcal{B}_i u_i = 0, \quad x \in \partial\Omega,\ t > 0, \tag{4.6}$$

where $m_i > 1$, the coefficients $\alpha_i(x,t)$ are continuous, ω-periodic in t and change sign, $a_i(x,t)$, $b_i(x,t)$ and $c_i(x,t)$ are positive and ω-periodic in t. System (4.3)–(4.5) with $m_i = 1$ has been investigated by many authors, see example [10,12,28]. In the case $u_3 \equiv 0$, system (4.3)–(4.5) reduces to the two species competition model. Our following result shows that in the case of $m_i > 1$ the effect of mutualist u_3 on the nature of the problem is weaker than that in the case of $m_i = 1$.

Theorem 4.2. *Assume that*

$$\left\{x \in \Omega;\ \frac{1}{\omega}\int_0^\omega \left(\alpha_1(x,t) - a_2(x,t)\max_{\Omega\times[0,\omega]}\big[b_2(x,t)\alpha_2(x,t)\big]\right)dt > 0\right\} \neq \emptyset, \tag{4.7}$$

$$\left\{x \in \Omega;\ \frac{1}{\omega}\int_0^\omega \left(\alpha_2(x,t) - b_1(x,t)\max_{\Omega\times[0,\omega]}\big[a_1(x,t)\alpha_1(x,t)\big]\right)dt > 0\right\} \neq \emptyset \tag{4.8}$$

and

$$\left\{x \in \Omega;\ \frac{1}{\omega}\int_0^\omega \alpha_3(x,t)\,dt > 0\right\} \neq \emptyset, \tag{4.9}$$

then problem (4.3)–(4.6) *admits two nonnegative nontrivial periodic solutions* $(\bar\theta_1, \underline{\theta}_2, \bar\theta_3)$ *and* $(\underline{\theta}_1, \bar\theta_2, \underline{\theta}_3)$ *with* $\underline{\theta}_i \leqslant \bar\theta_i$, $i = 1, 2, 3$.

Proof. By Definition 2.2, if ω-periodic functions $\tilde{u}_1$, $\hat{u}_2$ and $\tilde{u}_3$ satisfy

$$\frac{\partial \tilde{u}_1}{\partial t} - \Delta \tilde{u}_1^{m_1} \geqslant \tilde{u}_1\left(\alpha_1(x,t) - \frac{\tilde{u}_1}{a_1(x,t)} - \frac{a_2(x,t)\hat{u}_{2\tau}}{1+a_3(x,t)\tilde{u}_{3\tau}}\right), \quad x \in \Omega,\ t > 0, \tag{4.10}$$

$$\frac{\partial \hat{u}_2}{\partial t} - \Delta \hat{u}_2^{m_2} \leqslant \hat{u}_2\left(\alpha_2(x,t) - b_1(x,t)\tilde{u}_{1\tau} - \frac{\hat{u}_2}{b_2(x,t)}\right), \tag{4.11}$$

$$\frac{\partial \tilde{u}_3}{\partial t} - \Delta \tilde{u}_3^{m_3} \geqslant \tilde{u}_3\left(\alpha_3(x,t) - \frac{\tilde{u}_3}{c_0(x,t)+c_1(x,t)\tilde{u}_{1\tau}}\right), \tag{4.12}$$

$$\mathcal{B}_i\tilde{u}_i \geqslant 0 \quad (i = 1, 3), \qquad \mathcal{B}_i\hat{u}_2 \leqslant 0, \quad x \in \partial\Omega,\ t > 0, \tag{4.13}$$

and $\hat{u}_1, \tilde{u}_2, \hat{u}_3$ satisfy the corresponding reverse inequalities, then $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ and $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ are upper and lower periodic solutions of problem (4.3)–(4.6), respectively.

Define

$$\tilde u_1=\max_{\Omega\times[0,\omega]}\big[a_1(x,t)\alpha_1(x,t)\big],$$
$$\tilde u_2=\max_{\Omega\times[0,\omega]}\big[b_2(x,t)\alpha_2(x,t)\big]$$

and

$$\tilde u_3=\max_{\Omega\times[0,\omega]}\big[\big(c_0(x,t)+c_1(x,t)\tilde u_1(x,t)\big)\alpha_3(x,t)\big],$$

then by assumptions (4.7)–(4.9) and the argument in the proof of Theorem 4.1, we can construct $\hat u_i$ $(i=1,2,3)$ such that $(\tilde u_1,\tilde u_2,\tilde u_3)$ and $(\hat u_1,\hat u_2,\hat u_3)$ are the ordered periodic upper and lower solutions of problem (4.3)–(4.6).

In fact, denote

$$g_1(x,t)=\alpha_1(x,t)-a_2(x,t)\max_{\Omega\times[0,\omega]}\big[b_2(x,t)\alpha_2(x,t)\big],$$
$$g_2(x,t)=\alpha_2(x,t)-b_1(x,t)\max_{\Omega\times[0,\omega]}\big[a_1(x,t)\alpha_1(x,t)\big]$$

and

$$g_3(x,t)=\alpha_3(x,t),$$

then by assumptions (4.7)–(4.9), there exist a ball $B_{\rho_i}(x_i)\subset\Omega$ $(i=1,2,3)$ and $\delta>0$ such that $\frac{1}{\omega}\int_0^\omega g_i(x,t)\,dt\geqslant\delta$ for $x\in\overline{B_{\rho_i}(x_i)}$. Let $\underline{\beta}_i(t)=\min_{x\in B_{\rho_i}(x_i)}\{g_i(x,t)\}$, $\beta_i(t)=\underline{\beta}_i(t)-\frac{1}{\omega}\int_0^\omega\underline{\beta}_i(t)\,dt$ and

$$\hat u_i(x,t)=\begin{cases}\varepsilon\exp\big(\int_0^t\beta_i(s)\,ds\big)\psi_i^{\frac{1}{m_i}}(x), & (x,t)\in B_{\rho_i}(x_i)\times\mathbb{R}^+,\\ 0, & (x,t)\in(\Omega\setminus B_{\rho_i}(x_i))\times\mathbb{R}^+,\end{cases}$$

where $\psi_i(x)$ is the eigenfunction corresponding first eigenvalue μ_{i1} of the eigenvalue problem

$$-\Delta\psi=\mu\psi,\quad x\in B_{\rho_i}(x_i),$$
$$\psi=0,\quad x\in\partial B_{\rho_i}(x_i)$$

with $\|\psi_i\|_{L^\infty}=1$. Then it can be verified that $(\tilde u_1,\tilde u_2,\tilde u_3)$ and $(\hat u_1,\hat u_2,\hat u_3)$ are the ordered periodic upper and lower solutions of problem (4.3)–(4.6), provided that

$$\varepsilon<\min\left\{\min_{1\leqslant i\leqslant3}\left\{\left(\frac{\delta}{2\mu_{i1}}\right)^{\frac{1}{m_i-1}}\right\},\frac{\delta}{2}\min\Big\{\min_{\Omega\times[0,\omega]}a_1(x,t),\min_{\Omega\times[0,\omega]}b_2(x,t),\min_{\Omega\times[0,\omega]}c_0(x,t)\Big\},\min_{1\leqslant i\leqslant3}\{\tilde u_i\}\right\}\min_{1\leqslant i\leqslant3}\times\left\{\min_{t\in[0,\omega]}\exp\left(\int_t^0\beta_i(s)\,ds\right)\right\}.$$

Therefore by Theorem 3.2, problem (4.3)–(4.6) admits a pair of periodic quasi-solutions $(\underline{\theta}_1,\underline{\theta}_2,\underline{\theta}_3)$ and $(\bar\theta_1,\bar\theta_2,\bar\theta_3)$ with $\underline{\theta}_i\leqslant\bar\theta_i$ $(i=1,2,3)$, which satisfies

$$\frac{\partial\bar\theta_1}{\partial t}-\Delta\bar\theta_1^{m_1}=\bar\theta_1\left(\alpha_1(x,t)-\frac{\bar\theta_1}{a_1(x,t)}-\frac{a_2(x,t)\underline{\theta}_{2\tau}}{1+a_3(x,t)\bar\theta_{3\tau}}\right),\quad x\in\Omega,\ t>0,$$
$$\frac{\partial\underline{\theta}_2}{\partial t}-\Delta\underline{\theta}_2^{m_2}=\underline{\theta}_2\left(\alpha_2(x,t)-b_1(x,t)\bar\theta_{1\tau}-\frac{\underline{\theta}_2}{b_2(x,t)}\right),$$
$$\frac{\partial\bar\theta_3}{\partial t}-\Delta\bar\theta_3^{m_3}=\bar\theta_3\left(\alpha_3(x,t)-\frac{\bar\theta_3}{c_0(x,t)+c_1(x,t)\bar\theta_{1\tau}}\right),$$
$$\frac{\partial\underline{\theta}_1}{\partial t}-\Delta\underline{\theta}_1^{m_1}=\underline{\theta}_1\left(\alpha_1(x,t)-\frac{\underline{\theta}_1}{a_1(x,t)}-\frac{a_2(x,t)\bar\theta_{2\tau}}{1+a_3(x,t)\underline{\theta}_{3\tau}}\right),$$
$$\frac{\partial\bar\theta_2}{\partial t}-\Delta\bar\theta_2^{m_2}=\bar\theta_2\left(\alpha_2(x,t)-b_1(x,t)\underline{\theta}_{1\tau}-\frac{\bar\theta_2}{b_2(x,t)}\right),$$
$$\frac{\partial\underline{\theta}_3}{\partial t}-\Delta\underline{\theta}_3^{m_3}=\underline{\theta}_3\left(\alpha_3(x,t)-\frac{\underline{\theta}_3}{c_0(x,t)+c_1(x,t)\underline{\theta}_{1\tau}}\right),$$

and thus problem (4.3)–(4.6) has periodic solutions $(\underline{\theta}_1,\bar\theta_2,\underline{\theta}_3)$ and $(\bar\theta_1,\underline{\theta}_2,\bar\theta_3)$. $\square$

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